

SCHMITT–VOGEL TYPE LEMMA FOR REDUCTIONS

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ABSTRACT. The lemma given by Schmitt and Vogel is an important tool in the study of arithmetical rank of squarefree monomial ideals. In this paper, we give a Schmitt–Vogel type lemma for reductions as an analogous result.

0. INTRODUCTION

Throughout this paper, let R be a commutative Noetherian ring with non-zero identity. Let I be an ideal of R . Then the *arithmetical rank* of I is defined by

$$\text{ara } I := \min\{r : \text{there exist some } a_1, \dots, a_r \in R \text{ such that } \sqrt{(a_1, \dots, a_r)} = \sqrt{I}\}.$$

If $\sqrt{(a_1, \dots, a_r)} = \sqrt{I}$ holds, then we say that a_1, \dots, a_r generate I *up to radical*.

Assume that R is a polynomial ring over a field K and I is generated by squarefree monomials. Then we have the following inequalities:

$$\text{height } I \leq \text{pd}_R R/I = \text{cd}(I) \leq \text{ara } I \leq \mu(I),$$

where $\text{height } I$ (resp. $\text{pd}_R R/I$, $\text{cd}(I)$, $\mu(I)$) denotes the height of I (resp. the projective dimension of R/I over R , the cohomological dimension of I , the minimal number of generators of I); see e.g. [7]. Many researchers, e.g. Barile [1, 2, 3, 4, 5], Schmitt and Vogel [11] and the authors [7, 8] have proved $\text{ara } I = \text{pd}_R R/I$ using the following lemma given by Schmitt and Vogel [11] or its generalizations.

Fact (Schmitt and Vogel [11, Lemma, p. 249]). *Let \mathcal{P} be a finite subset of R , and let I be the ideal generated by \mathcal{P} . Let $r \geq 0$ be an integer. Assume that there exist subsets $\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_r$ of \mathcal{P} such that the following conditions are satisfied:*

- (i) $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1 \cup \dots \cup \mathcal{P}_r$.
- (ii) $\#\mathcal{P}_0 = 1$.
- (iii) *For each ℓ ($0 < \ell \leq r$) and for every $a, a'' \in \mathcal{P}_\ell$ with $a \neq a''$, there exist an integer ℓ' ($0 \leq \ell' < \ell$), and elements $a' \in \mathcal{P}_{\ell'}$, such that $aa'' \in (a')$.*

If we set

$$g_\ell = \sum_{a \in \mathcal{P}_\ell} a, \quad \ell = 0, 1, \dots, r,$$

then $\sqrt{I} = \sqrt{(g_0, g_1, \dots, g_r)}$.

An ideal $J \subset I$ is said to be a *reduction* if there exists some integer $s \geq 1$ such that $I^{s+1} = JI^s$ holds. When this is the case, $\sqrt{J} = \sqrt{I}$ holds. If J is minimal among reductions of I with respect to inclusion, then it is said to be a *minimal reduction* of I . Let R be a polynomial ring over a field K and I a homogeneous ideal of R ,

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or let R be a local ring with unique maximal ideal \mathfrak{m} and $K = R/\mathfrak{m}$ and I an ideal of R . If K is infinite, then for any (homogeneous) ideal I , we can take a minimal reduction J of I and the minimal number of generators of J is independent of the choice of J ; see [9]. The number of generators of J is called the *analytic spread* of I (denoted by $\ell(I)$) and it gives an upper bound for $\text{ara } I$. In the commutative ring theory, the minimal reduction plays an important role because it admits the same integral closure as the original ideal. Moreover, the analytic spread is equal to the Krull dimension of the fiber cone $F(I) = \bigoplus_{n \geq 0} I^n/\mathfrak{m}I^n$ of I in a local ring (R, \mathfrak{m}) , and hence it is an important invariant.

The main purpose of this note is to give an analogous result of the lemma due to Schmitt and Vogel [11, Lemma, p. 249] for reductions; see Theorem 1.1. For instance, let us consider the following monomial ideal in a suitable polynomial ring R :

$$(0.1) \quad I = (x_{11}, \dots, x_{1h_1}) \cap \dots \cap (x_{q1}, \dots, x_{qh_q}).$$

In order to give an upper bound for $\text{cd}(I)$, Schenzel and Vogel [10] computed $\text{depth } R/I^\ell$ for all $\ell \geq 1$, and proved

$$\text{cd}(I) \leq \ell(I) \leq \text{depth } R - \inf_{\ell} \text{depth } R/I^\ell = \sum_{i=1}^q h_i - q + 1 \quad (= \text{pd}_R R/I),$$

where the second inequality is known as Burch's inequality. On the other hand, Schmitt and Vogel [11] constructed $\text{pd}_R R/I$ generators up to radical using their lemma. By using Theorem 1.1 instead of their lemma, we can provide a minimal reduction with $\text{pd}_R R/I$ generators; see Example 1.3.

In Section 2, we prove the main theorem. In order to do that, we give analogous results (see Theorems 2.1, 2.5) of generalizations of the lemma due to Barile for reductions, and prove them.

1. SCHMITT–VOGEL TYPE LEMMA FOR REDUCTIONS AND ITS APPLICATION

The following theorem is the main result in this paper, which gives an analogous result of [11, Lemma, p. 249] proved by Schmitt and Vogel. Note that the theorem immediately follows from Theorem 2.1, which will be proved in Section 2.

Theorem 1.1 (Schmitt–Vogel type lemma for reductions). *Let \mathcal{P} be a finite subset of R , and let I be the ideal generated by \mathcal{P} . Let $r \geq 0$ be an integer. Assume that there exist subsets $\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_r$ of \mathcal{P} such that the following conditions are satisfied:*

$$(SV1) \quad \mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1 \cup \dots \cup \mathcal{P}_r.$$

$$(SV2) \quad \sharp \mathcal{P}_0 = 1.$$

$$(SV3) \quad \text{For each } \ell \ (0 < \ell \leq r) \text{ and for every } a, a'' \in \mathcal{P}_\ell \text{ with } a \neq a'', \text{ there exist an integer } \ell' \ (0 \leq \ell' < \ell), \text{ and elements } a' \in \mathcal{P}_{\ell'}, b \in I \text{ such that } aa'' = a'b.$$

If we set

$$g_\ell = \sum_{a \in \mathcal{P}_\ell} a, \quad \ell = 0, 1, \dots, r,$$

then $J = (g_0, g_1, \dots, g_r)$ is a reduction of I .

We now restrict our attention to the following case: R is a polynomial ring over a field K and I is a squarefree monomial ideal of R . In this case, as an application of the above theorem, we have the following result.

Corollary 1.2. *Let R be a polynomial ring and I a squarefree monomial ideal of R . Assume that there exist finite subsets $\mathcal{P}_0, \dots, \mathcal{P}_r$ of R satisfying the assumptions in Theorem 1.1 for $r = \text{pd}_R R/I - 1$. Then (g_0, g_1, \dots, g_r) is a minimal reduction of I , and $\ell(I) = \text{ara } I = \text{pd}_R R/I = r + 1$.*

Proof. Since I is a squarefree monomial ideal, we have

$$r + 1 = \text{pd}_R R/I = \text{cd}(I) \leq \text{ara } I \leq \ell(I).$$

On the other hand, Theorem 1.1 implies $\ell(I) \leq r + 1$. Hence we get the desired assertion. \square

We can apply our results to Alexander dual of complete intersection monomial ideals; see below.

Example 1.3 (Alexander dual of complete intersection monomial ideals).

Let $I \subseteq R$ be a squarefree monomial ideal of the following shape:

$$(1.1) \quad (x_{11}, \dots, x_{1h_1}) \cap \dots \cap (x_{q1}, \dots, x_{qh_q}),$$

where $R = K[x_{11}, \dots, x_{1h_1}, \dots, x_{q1}, \dots, x_{qh_q}]$ is a polynomial ring over a field K . Note that I can be regarded as the Alexander dual of complete intersection monomial ideal $(x_{11} \cdots x_{1h_1}, \dots, x_{q1} \cdots x_{qh_q})$ if $h_1, \dots, h_q \geq 2$.

Set $r = h_1 + \dots + h_q - q$ and

$$g_\ell = \sum_{\ell_1 + \dots + \ell_q = \ell} x_{1\ell_1} x_{2\ell_2} \cdots x_{q\ell_q}, \quad \ell = 0, 1, \dots, r.$$

Then (g_0, g_1, \dots, g_r) is a minimal reduction of I . In particular,

$$\ell(I) = \text{ara } I = \text{pd}_R R/I = \sum_{i=1}^q h_i - q + 1.$$

Proof. It is known that

$$r + 1 = \text{pd}_R R/I = \text{ara } I \leq \ell(I);$$

see e.g. [11, Theorem] or [7, Section 5].

For each $\ell = 0, 1, \dots, r$, we set

$$\mathcal{P}_\ell = \{x_{1\ell_1} \cdots x_{q\ell_q} : 1 \leq \ell_j \leq h_j, \ell_1 + \dots + \ell_q = \ell + q\}.$$

Then I is generated by all monomials in $\mathcal{P}_0 \cup \dots \cup \mathcal{P}_r$, and \mathcal{P}_0 consists of only one element $x_{11} \cdots x_{q1}$. Thus it suffices to show that if $a, a'' \in \mathcal{P}_\ell$ with $a \neq a''$ then there exist $a' \in \mathcal{P}_{\ell'}$ for some $\ell' < \ell$ and $b \in I$ such that $aa'' = a'b$. Write

$$a = x_{1i_1} x_{2i_2} \cdots x_{qi_q}, \quad a'' = x_{1j_1} x_{2j_2} \cdots x_{qj_q},$$

where $i_1 + \dots + i_q = j_1 + \dots + j_q = \ell + q$. As $a \neq a''$, there exists an integer k ($1 \leq k \leq q$) such that $i_k > j_k$. We may assume that $k = 1$ without loss of generality.

Then if we set

$$a' = a \cdot \frac{x_{1j_1}}{x_{1i_1}} = x_{1j_1}x_{2i_2} \cdots x_{qi_q}, \quad b = a'' \cdot \frac{x_{1i_1}}{x_{1j_1}} = x_{1i_1}x_{2j_2} \cdots x_{qj_q} \in I,$$

then $aa'' = a'b$ and $a' \in \mathcal{P}_{\ell'}$, where

$$\ell' = j_1 + i_2 + \cdots + i_q - q < i_1 + i_2 + \cdots + i_q - q = \ell.$$

Hence we can apply Corollary 1.2. \square

Remark 1.4. We use the same notation as in Example 1.3. Schmitt and Vogel [11] proved $\text{ara } I = \text{pd}_R R/I$ by showing $\sqrt{(g_0, g_1, \dots, g_r)} = \sqrt{I}$. Thus the above example gives an improvement of their result.

We can generalize Example 1.3 as follows.

Proposition 1.5. *For each $i = 1, 2, \dots, s$, let I_i be a squarefree monomial ideal of the shape (1.1):*

$$I_i = (x_{11}^{(i)}, \dots, x_{1h_1^{(i)}}^{(i)}) \cap \cdots \cap (x_{q^{(i)}1}^{(i)}, \dots, x_{q^{(i)}h_{q^{(i)}}^{(i)}}^{(i)}).$$

Let $G(I_i)$ be the minimal set of monomial generators of I_i . Suppose that there are no variables which appear in both $G(I_i)$ and $G(I_j)$ for each i, j with $i \neq j$. For I_i , set $g_\ell^{(i)}$ as in Example 1.3. Then

$$(g_\ell^{(i)} : i = 1, \dots, s, \ell = 0, 1, \dots, h_1^{(i)} + \cdots + h_{q^{(i)}}^{(i)} - q^{(i)})$$

is a minimal reduction of $I_1 + \cdots + I_s$. In particular, $\ell(I_1 + \cdots + I_s) = \ell(I_1) + \cdots + \ell(I_s)$.

In order to prove Proposition 1.5, it is enough to show the following lemma.

Lemma 1.6. *Let R, S be polynomial rings over a field K with no common variables, and put $T = R \otimes_K S$. Let $I \subseteq R$ (resp. $J \subseteq S$) be a squarefree monomial ideal. Then:*

- (1) $\text{pd}_T T/(IT + JT) = \text{pd}_R R/I + \text{pd}_S S/J$.
- (2) *Assume that $\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_r \subseteq R$ (resp. $\mathcal{Q}_0, \mathcal{Q}_1, \dots, \mathcal{Q}_s \subseteq S$) satisfies (SV1), (SV2) and (SV3) in Theorem 1.1. Then $\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_r, \mathcal{Q}_0, \mathcal{Q}_1, \dots, \mathcal{Q}_s$ also satisfies the same conditions as finite subsets of T .*

Proof. (1) Let F_\bullet (resp. G_\bullet) be a minimal free resolution of R/I over R (resp. S/J over S). Then $F_\bullet \otimes_K G_\bullet$ is a minimal free resolution of $T/(IT + JT)$. Thus we have $\text{pd}_T T/(IT + JT) = \text{pd}_R R/I + \text{pd}_S S/J$.

- (2) It is clear by definition. \square

Remark 1.7. We use the same notation as in Lemma 1.6. Then it is easy to see that $\text{ara}(IT + JT) \leq \text{ara } I + \text{ara } J$ holds. If both $\text{ara } I = \text{pd}_R R/I$ and $\text{ara } J = \text{pd}_S S/J$ hold, then the equality holds. But we do *not* know whether it is always true. Moreover, it seems that a similar result holds for analytic spreads, but we do *not* have any proof in general.

2. PROOF OF THE THEOREM

In this section, we prove Theorem 1.1, which is an analogous result of the lemma by Schmitt–Vogel for reductions. But the lemma has been generalized by Barile [1, 3]; see also [5] for another version. So, in this section, we prove analogous results for two generalizations by Barile; see Theorems 2.1, 2.5.

The following theorem gives an analogous result for Barile [3, Lemma 2.1], which is a generalization of Theorem 1.1.

Theorem 2.1. *Let $\mathcal{P} \subset R$ be a finite subset, and put $I = (\mathcal{P})$. Let $\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_r$ be subsets of \mathcal{P} . Assume that the following conditions:*

- (B1) $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1 \cup \dots \cup \mathcal{P}_r$.
- (B2) $\sharp \mathcal{P}_0 = 1$.
- (B3) *For each ℓ ($0 < \ell \leq r$) and for every $a, a'' \in \mathcal{P}_\ell$ with $a \neq a''$, there exists an integer $m \geq 1$ such that $(aa'')^m \in (\mathcal{P}_0 \cup \dots \cup \mathcal{P}_{\ell-1})I^{2m-1}$.*

Set

$$g_\ell = \sum_{a \in \mathcal{P}_\ell} a, \quad \ell = 0, 1, \dots, r.$$

Then $J = (g_0, g_1, \dots, g_r)$ is a reduction of I .

Remark 2.2. The difference between Theorem 2.1 and the original result of Barile [3] is in the condition (B3). The condition of the original result corresponding to (B3) is

- (B3)' For each ℓ ($0 < \ell \leq r$) and for every $a, a'' \in \mathcal{P}_\ell$ with $a \neq a''$, there exists an integer $m \geq 1$ such that $(aa'')^m \in (\mathcal{P}_0 \cup \dots \cup \mathcal{P}_{\ell-1})$.

Proof of Theorem 2.1. Since $J \subseteq I$, it suffices to show $I^{s+1} \subset JI^s$. In order to do that, we set $\sharp \mathcal{P}_\ell = c_\ell$ and $I_\ell = (\mathcal{P}_0 \cup \dots \cup \mathcal{P}_\ell)$ for each $\ell = 0, 1, \dots, r$. Moreover, for each ℓ , we can choose an integer $m_\ell \geq 1$ such that

$$(aa'')^{m_\ell} \in I_{\ell-1}I^{2m_\ell-1}$$

for all $a, a'' \in \mathcal{P}_\ell$ with $a \neq a''$ by assumption. Then it is enough to prove

$$(2.1) \quad I_j^{c_1 \dots c_j m_1 \dots m_j} \subset I_{j-1}^{c_1 \dots c_{j-1} m_1 \dots m_{j-1}} I^{(c_1 \dots c_{j-1} m_1 \dots m_{j-1})(c_j m_j - 1)} + JI^{c_1 \dots c_j m_1 \dots m_j - 1}$$

for each $j = 0, 1, \dots, r$.

The case of $j = 0$ is clear because $I_0 = (\mathcal{P}_0) = (g_0) \subset J$.

Now suppose $j = \ell \geq 1$ and assume that (2.1) holds for every $j \leq \ell - 1$. To prove (2.1) for $j = \ell$, it is enough to show that for arbitrary $c_1 \dots c_\ell m_1 \dots m_\ell$ elements (to take the same elements is allowed) in $\mathcal{P}_0 \cup \dots \cup \mathcal{P}_\ell$, the product of all elements is contained in the right hand side of (2.1). We divide these elements into $c_1 \dots c_{\ell-1} m_1 \dots m_{\ell-1}$ sequences of $c_\ell m_\ell$ elements, and show that the product of the elements in each sequence is in $I_{\ell-1}I^{c_\ell m_\ell - 1} + JI^{c_\ell m_\ell - 1}$.

In what follows, we discuss about only one sequence. If there exists an element of $\mathcal{P}_0 \cup \dots \cup \mathcal{P}_{\ell-1}$ in the sequence, then it is clear that the product is in $I_{\ell-1}I^{c_\ell m_\ell - 1}$. Therefore, we may assume that all elements in the sequence are in \mathcal{P}_ℓ . If we can find a pair (a, a'') with $a \neq a''$ which appear at least m_ℓ times in this sequence, then the assumption (B3) yields that the product of all elements in the sequence

is contained in $I_{\ell-1}I^{c_\ell m_\ell-1}$. Otherwise, we pick up an element a_1 the number of times (say, d) which appears in the sequence is maximal. Note that $d > m_\ell$. Let $\mathcal{P}_\ell = \{a_1, a_2, \dots, a_{c_\ell}\}$. Then the product of all elements in the sequence is

$$\begin{aligned}
a_1^d a_2^{k_2} \cdots a_{c_\ell}^{k_{c_\ell}} &= a_1^{m_\ell} a_1^{d-m_\ell} a_2^{k_2} \cdots a_{c_\ell}^{k_{c_\ell}} \\
&= a_1^{m_\ell} \left(g_\ell - \sum_{i=2}^{c_\ell} a_i \right)^{d-m_\ell} a_2^{k_2} \cdots a_{c_\ell}^{k_{c_\ell}} \\
&= g_\ell \cdot (\text{the products of } c_\ell m_\ell - 1 \text{ elements of } \mathcal{P}_\ell) \\
&\quad + a_1^{m_\ell} \left(\sum_{i=2}^{c_\ell} a_i \right)^{d-m_\ell} a_2^{k_2} \cdots a_{c_\ell}^{k_{c_\ell}} \\
&= g_\ell \cdot (\text{the products of } c_\ell m_\ell - 1 \text{ elements of } \mathcal{P}_\ell) \\
&\quad + \sum a_1^{m_\ell} a_2^{k'_2} \cdots a_{c_\ell}^{k'_{c_\ell}},
\end{aligned}$$

where $k_2 + \dots + k_{c_\ell} = c_\ell m_\ell - d$ and $k'_2 + \dots + k'_{c_\ell} = (c_\ell - 1)m_\ell$. Then there exists an integer j with $2 \leq j \leq c_\ell$ such that $k'_j \geq m_\ell$. By a similar argument as above, the right-hand side is contained in $JI^{c_\ell m_\ell-1} + I_{\ell-1}I^{c_\ell m_\ell-1}$. Hence we have finished the proof. \square

Proof of Theorem 1.1. Assume that I satisfies (SV1), (SV2), and (SV3). Then it also satisfies (B1), (B2) and (B3). Hence the assertion immediately follows from Theorem 2.1. \square

In the proof of the following two examples, we need Theorem 2.1 instead of Theorem 1.1.

Example 2.3. Let K be a field, and let $m \geq 2$ be an integer. Consider the hypersurface $R = K[[x, y, z]]/(x^m y^m - z^{2m})$. For an ideal $I = (x, y, z)R$, we put

$$\mathcal{P}_0 = \{z\}, \quad \mathcal{P}_1 = \{x, y\}.$$

Then since $(xy)^m = z \cdot z^{2m-1} \in (\mathcal{P}_0)I^{2m-1}$, we can conclude that $x + y, z$ is a (minimal) reduction by virtue of Theorem 2.1. But we cannot apply Theorem 1.1 to this case because $xy \notin (z)$.

Example 2.4. Let $R = K[x_1, x_2, x_3, x_4, x_5, x_6]$ be a polynomial ring over a field K . For an ideal

$$I = (x_1 x_2 + x_1 x_3, x_1 x_4, x_1 x_5, x_1 x_6, x_2 x_5, x_2 x_6, x_3 x_4, x_3 x_6, x_4 x_5, x_4 x_6, x_5 x_6),$$

we put

$$\begin{aligned}
\mathcal{P}_0 &= \{x_1 x_6\}, & \mathcal{P}_1 &= \{x_1 x_5, x_2 x_6\}, \\
\mathcal{P}_2 &= \{x_1 x_4, x_3 x_6\}, & \mathcal{P}_3 &= \{x_2 x_5, x_4 x_6\}, \\
\mathcal{P}_4 &= \{x_3 x_4, x_5 x_6\}, & \mathcal{P}_5 &= \{x_1 x_2 + x_1 x_3, x_4 x_5\}.
\end{aligned}$$

Then we can conclude that

$$J = (x_1 x_6, x_1 x_5 + x_2 x_6, x_1 x_4 + x_3 x_6, x_2 x_5 + x_4 x_6, x_3 x_4 + x_5 x_6, x_1 x_2 + x_1 x_3 + x_4 x_5)$$

is a (minimal) reduction of I by Theorem 2.1. But we cannot apply Theorem 1.1 because the product of $(x_1x_2 + x_1x_3) \in \mathcal{P}_5$ and $x_4x_5 \in \mathcal{P}_5$ is not contained in the ideal (a') for any element $a' \in \mathcal{P}_0 \cup \dots \cup \mathcal{P}_4$.

Next, we refine the result by Barile [1, Proposition 1.1]

Theorem 2.5. *Assume that R is a local ring. Let $\mathcal{P} \subset R$ be a finite subset, and let $\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_r$ be subsets of \mathcal{P} . We set $\sharp\mathcal{P}_\ell = c_\ell$ for all ℓ and $I = (\mathcal{P})$. Assume that the following conditions are satisfied:*

(Ba1) $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1 \cup \dots \cup \mathcal{P}_r$.

(Ba2) $\sharp\mathcal{P}_0 = 1$.

(Ba3) *For each ℓ ($0 < \ell \leq r$) with $c_\ell \geq 2$, there exists an integer n_ℓ with $2 \leq n_\ell \leq c_\ell$ such that for arbitrary n_ℓ distinct elements $p_1, p_2, \dots, p_{n_\ell} \in \mathcal{P}_\ell$, there exist an integer ℓ' with $0 \leq \ell' < \ell$, elements $p' \in \mathcal{P}_{\ell'}$ and $b \in I^{n_\ell-1}$ such that $p_1p_2 \cdots p_{n_\ell} = p'b$.*

For $0 \leq \ell \leq r$ with $c_\ell = 1$, we set $n_\ell = 2$. For each $\ell = 0, 1, \dots, r$, let $A^{(\ell)} = (a_{ij}^{(\ell)})$ be an $(n_\ell - 1) \times c_\ell$ matrix with $a_{ij}^{(\ell)} \in R$. Assume that all maximal minors of $A^{(\ell)}$ are unit in R . Set

$$\begin{aligned} \mathcal{P}_\ell &= \{p_1^{(\ell)}, p_2^{(\ell)}, \dots, p_{c_\ell}^{(\ell)}\}, \quad 0 \leq \ell \leq r, \\ g_i^{(\ell)} &= \sum_{j=1}^{c_\ell} a_{ij}^{(\ell)} p_j^{(\ell)}, \quad 1 \leq i \leq n_\ell - 1, \quad 0 \leq \ell \leq r, \\ J &= (g_i^{(\ell)} : 0 \leq \ell \leq r, 1 \leq i \leq n_\ell - 1). \end{aligned}$$

Then J is a reduction of I .

Remark 2.6. The difference between Theorem 2.5 and the original result of Barile [1] is in the condition (Ba3). The condition of the original result corresponding to (Ba3) is

(Ba3)' For each ℓ ($0 < \ell \leq r$) with $c_\ell \geq 2$, there exists some integer n_ℓ , $2 \leq n_\ell \leq c_\ell$ such that for arbitrary n_ℓ distinct elements $p_1, p_2, \dots, p_{n_\ell} \in \mathcal{P}_\ell$, there exist ℓ' with $0 \leq \ell' < \ell$ and $p' \in \mathcal{P}_{\ell'}$, such that $p_1p_2 \cdots p_{n_\ell} \in (p')$.

Proof of Theorem 2.5. It is enough to show $I^{s+1} \subset JI^s$ for some $s \geq 0$.

For each $\ell = 0, 1, \dots, r$, we set $I_\ell = (\mathcal{P}_0 \cup \dots \cup \mathcal{P}_\ell)$. Then it is enough to prove

$$(2.2) \quad I_j^{n_0n_1 \cdots n_j} \subset I_{j-1}^{n_0n_1 \cdots n_{j-1}} I^{(n_0n_1 \cdots n_{j-1})(n_j-1)} + JI^{n_0n_1 \cdots n_j-1}$$

for each $j = 0, 1, \dots, r$.

The case of $j = 0$ is clear because $p_0 = g_0 \in J$ by the assumption (Ba2).

Now suppose $j = \ell \geq 1$ and assume that (2.2) holds for every $j \leq \ell - 1$. In order to prove (2.2) for $j = \ell$, it is enough to show that for arbitrary $n_0n_1 \cdots n_\ell$ elements (to take the same elements is allowed) in $\mathcal{P}_0 \cup \dots \cup \mathcal{P}_\ell$, the product of these elements is contained in the right hand side of (2.2). We divide these elements into $n_0n_1 \cdots n_{\ell-1}$ sequences of n_ℓ elements, and show that the product of all elements in each sequence is contained in $I_{\ell-1}I^{n_\ell-1} + JI^{n_\ell-1}$.

In what follows, we discuss about only one sequence. If there exists an element of $\mathcal{P}_0 \cup \dots \cup \mathcal{P}_{\ell-1}$ in the sequence, then it is clear that the product is contained in

$I_{\ell-1}I^{n_{\ell}-1}$. Therefore, we may assume that all elements in the sequence belong to \mathcal{P}_{ℓ} .

In the following, we omit the symbol ℓ for simplicity. Consider the product

$$\mu = p_1^{k_1} p_2^{k_2} \cdots p_c^{k_c}, \quad k_1 + k_2 + \cdots + k_c = n, \quad k_i \geq 0$$

and set

$$t := t(\mu) := \#\{i : k_i = 1\}.$$

We prove $\mu \in I_{\ell-1}I^{n_{\ell}-1}$ by descending induction on t ($0 \leq t \leq n$).

If $t = n$, then μ is a product of distinct n elements in \mathcal{P}_{ℓ} . It follows that $\mu \in I_{\ell-1}I^{n_{\ell}-1}$ by the assumption (Ba3).

Now we consider the case where $0 \leq t \leq n-1$. Then we can assume without loss of generality that $k_1 = k_2 = \cdots = k_t = 1$ and $k_i \geq 2$ for any $i > t$. Notice that $t \leq n-2$. Let A' be the $(n-1) \times (n-1)$ submatrix of A consists of first $n-1$ columns of A . By assumption, A' is invertible. Since R is local, we may assume that it is possible to transform the matrix A to the matrix $B = (b_{ij})$ having the same size as A with $b_{ij} = \delta_{ij}$ for $1 \leq i \leq n-1$, $1 \leq j \leq n-1$ by elementary row operations. Then we put

$$g'_{t+1} = p_{t+1} + \sum_{j=t+2}^c b_{t+1j} p_j \in J.$$

Since $k_{t+1} \geq 2$, we have

$$\begin{aligned} \mu &= p_1 p_2 \cdots p_t p_{t+1} \left(g'_{t+1} - \sum_{j=t+2}^c b_{t+1j} p_j \right)^{k_{t+1}-1} p_{t+2}^{k_{t+2}} \cdots p_n^{k_n} \\ &= g'_{t+1} (\text{an element of } I^{n-1}) + p_1 p_2 \cdots p_t p_{t+1} \left(- \sum_{j=t+2}^c b_{t+1j} p_j \right)^{k_{t+1}-1} p_{t+2}^{k_{t+2}} \cdots p_n^{k_n} \\ &= (\text{an element of } JI^{n-1}) + \sum (\text{an element of } R) \cdot p_1 p_2 \cdots p_t p_{t+1} p_{t+2}^{k'_{t+2}} \cdots p_n^{k'_n}, \end{aligned}$$

where

$$t+1 + k'_{t+2} + \cdots + k'_n = t + k_{t+1} + k_{t+2} + \cdots + k_n = n.$$

Then the induction hypothesis implies that the second term in the last equation is contained in $I_{\ell-1}I^{n-1} + JI^{n-1}$. This completes the proof. \square

In the next example, the analytic spread of I is known, but we can provide a concrete minimal reduction using Theorem 2.5.

Example 2.7. Let $r \geq 2$ be an integer. Set $I = (x_1 x_2, x_2 x_3, \dots, x_{2r-1} x_{2r}, x_{2r} x_1)$, the edge ideal of the $2r$ -cycle ($r \geq 2$). Put

$$\begin{aligned} \mathcal{P}_{\ell} &= \{x_{2\ell+1} x_{2\ell+2}\}, \quad \ell = 0, 1, \dots, s-1, \\ \mathcal{P}_r &= \{x_2 x_3, x_4 x_5, \dots, x_{2r-2} x_{2r-1}, x_{2r} x_1\}. \end{aligned}$$

Then the assumptions of Theorem 2.5 are satisfied with $n_\ell = 2$ for $\ell = 0, 1, \dots, r-1$ and $n_r = r$. Moreover, since all maximal minors of the matrix

$$A^{(r)} = \begin{pmatrix} 1 & & & 1 \\ & 1 & & 1 \\ & & \ddots & \vdots \\ & & & 1 & 1 \end{pmatrix}$$

are unit in R , we obtain that

$$x_1x_2, x_3x_4, \dots, x_{2r-1}x_{2r}, x_2x_3 + x_{2r}x_1, x_4x_5 + x_{2r}x_1, \dots, x_{2r-2}x_{2r-1} + x_{2r}x_1$$

is a reduction of I by Theorem 2.5.

On the other hand, we have $\ell(I) = 2r - 1$ due to Vasconcelos [13] because any $2r$ -cycle is a bipartite graph. In particular, the above reduction is a minimal reduction of I .

In the following example, we cannot apply the above theorem, but we can find a minimal reduction by a similar argument as in the proof.

Example 2.8. Let $R = K[x_1, x_2, x_3, x_4, x_5]$ be a polynomial ring over an infinite field K , and let $a, b, c, d \in K \setminus \{0\}$ be distinct elements with each other. Let I be the edge ideal of the complete graph K_5 , that is, I is the ideal generated by the following squarefree monomials of degree 2:

$$x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_2x_3, x_2x_4, x_2x_5, x_3x_4, x_3x_5, x_4x_5.$$

Set

$$\begin{aligned} \mathcal{P}_0 &= \{x_1x_2\}, & \mathcal{P}_1 &= \{x_2x_3, x_4x_5\}, \\ \mathcal{P}_2 &= \{x_3x_4, x_1x_5\}, & \mathcal{P}_3 &= \{x_1x_3, x_1x_4, x_2x_4, x_2x_5, x_3x_5\}, \end{aligned}$$

and $I_\ell = (\mathcal{P}_0 \cup \dots \cup \mathcal{P}_\ell)$ for each $\ell = 0, 1, 2$. If we put

$$\begin{aligned} g_0 &= x_1x_2, \\ g_1 &= x_2x_3 + x_4x_5, \\ g_2 &= x_3x_4 + x_1x_5, \\ g_3 &= x_1x_3 + ax_1x_4 + bx_2x_4 + cx_2x_5 + dx_3x_5, \\ g_4 &= x_1x_3 + a^2x_1x_4 + b^2x_2x_4 + c^2x_2x_5 + d^2x_3x_5, \end{aligned}$$

then $J = (g_0, g_1, g_2, g_3, g_4)$ is a (minimal) reduction of I by a similar argument as in the proof of Theorem 2.5. Indeed, we note that $I_2^3 \subseteq (g_0, g_1, g_2)I^2$.

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REFERENCES

1. M. Barile, *On the number of equations defining certain varieties*, Manuscripta Math. **91** (1996), 483–494.
2. M. Barile, *On ideals whose radical is a monomial ideal*, Comm. Algebra **33** (2005), 4479–4490.
3. M. Barile, *On ideals generated by monomials and one binomial*, Algebra Colloq. **14** (2007), 631–638.
4. M. Barile, *On the arithmetical rank of an intersection of ideals*, Algebra Colloq. **15** (2008), 681–688.

5. M. Barile, *On the arithmetical rank of certain monomial ideals*, preprint, math.AC/0611790v2.
6. J. Herzog, T. Hibi, *The depth of powers of an ideal*, J. Algebra **291** (2005), 534–550.
7. K. Kimura, N. Terai and K. Yoshida, *Arithmetical rank of squarefree monomial ideals of small arithmetic degree*, J. Algebraic Combin. **29** (2009), 389–404.
8. K. Kimura, N. Terai and K. Yoshida, *Arithmetical rank of squarefree monomial ideals of deviation two*, to appear in: Proceedings of the Conference on Combinatorial Commutative Algebra and Computer Algebra (V. Ene and E. Miller eds.).
9. D. G. Northcott and D. Rees, *Reductions of ideal in local rings*, Proc. Camb. Philos. Soc. **50** (1954), 145–158.
10. P. Schenzel and W. Vogel, *On set-theoretic intersections*, J. Algebra **48** (1977), 401–408.
11. T. Schmitt and W. Vogel, *Note on set-theoretic intersections of subvarieties of projective space*, Math. Ann. **245** (1979), 247–253.
12. A. Simis, W.V. Vasconcelos and R.H. Villarreal, *On the ideal theory of graphs*, J. Algebra **167** (1994), 389–416.
13. W.V. Vasconcelos, *Integral closure*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2005.

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